

Modular forms, modular symbols

(PARI-GP version 2.9.0)

Modular Forms

To be completed later.

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$. We let $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$; an element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \rightarrow$ the path from a to b . A path is coded by the pair $[a, b]$, where a, b are rationals or ∞ , denoting the point at infinity $(1 : 0)$.

Let $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(G), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k -valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either \mathbf{M}_k ($\varepsilon = 0$) or its \pm -parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$ -generators (g_i) of Δ .

| | |
|--|---|
| initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$ | <code>msinit($N, k, \{\varepsilon = 0\}$)</code> |
| the level M | <code>msgetlevel(M)</code> |
| the weight k | <code>msgetweight(M)</code> |
| the sign ε | <code>msgetsign(M)</code> |
| $\mathbf{Z}[G]$ -generators and relations for Δ | <code>mspathgens(M)</code> |
| Decompose $p = [a, b]$ on the (g_i) | <code>mspathlog(M, p)</code> |

Create a symbol

| | |
|--|--|
| Eisenstein symbol attached to cusp c | <code>msfromcusp(M, c)</code> |
| Cuspidal symbol attached to E/\mathbf{Q} | <code>msfromell(E)</code> |
| symbol having given Hecke eigenvalues | <code>msfromhecke($M, v, \{H\}$)</code> |
| is s a symbol ? | <code>msissymbol(M, s)</code> |
| the list of all $s(g_i)$ | <code>mseval(M, s)</code> |
| evaluate symbol s on path $p = [a, b]$ | <code>mseval(M, s, p)</code> |

Operators

An operator is given by a matrix of a fixed \mathbf{Q} -basis. H , if given, is a stable \mathbf{Q} -subspace of $\mathbf{M}_k(G)$: operator is restricted to H .

| | |
|---|--|
| matrix of Hecke operator T_p or U_p | <code>mshecke($M, p, \{H\}$)</code> |
| matrix of Atkin-Lehner w_Q | <code>msatkinlehner($M, Q, \{H\}$)</code> |
| matrix of the $*$ involution | <code>msstar($M, \{H\}$)</code> |

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a \mathbf{Q} -basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

| | |
|---|---|
| cuspidal subspace $S_k(G)^\varepsilon$ | <code>mscuspidal(M)</code> |
| Eisenstein subspace $E_k(G)^\varepsilon$ | <code>mseisenstein(M)</code> |
| new part of $S_k(G)^\varepsilon$ | <code>msnew(M)</code> |
| split H into simple subspaces (of $\dim \leq d$) | <code>mssplit($M, H, \{d\}$)</code> |
| (a_1, \dots, a_B) for attached newform | <code>msqexpansion($M, H, \{B\}$)</code> |

Overconvergent symbols and p -adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with non-zero eigenvalue a_p , we can attach a p -adic L -function L_p . The function L_p is defined on continuous characters of $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle \chi \rangle^{s_1} \tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ , taking values in spaces of p -adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* = 0 (fastest), and that $v_p(a_p) \geq \textit{flag}$ otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to Φ allowing to compute L_p to high accuracy.

| | |
|---|---|
| initialize Mp to lift symbols | <code>mspadicinit($M, p, n, \{\textit{flag}\}$)</code> |
| lift symbol ϕ | <code>mstooms(Mp, ϕ)</code> |
| eval overconvergent symbol Φ on path p | <code>msomseval(Mp, Φ, p)</code> |
| μ for p -adic L -functions | <code>mspadicmoments($Mp, S, \{D = 1\}$)</code> |
| $L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$ | <code>mspadicL($\mu, \{s = 0\}, \{r = 0\}$)</code> |
| $\hat{L}_p(\tau^i)(x)$ | <code>mspadicseries($\mu, \{i = 0\}$)</code> |

Based on an earlier version by Joseph H. Silverman
August 2016 v2.30. Copyright © 2016 K. Belabas
Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.
Send comments and corrections to Karim.Belabas@math.u-bordeaux.fr